Vertex Equitable Labeling of Vertex Signed Graph

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Abstract

A signed graph (or sigraph) $S=(G, \sigma)$ consists of an underlying graph $G=(V,E)$ and a function $\sigma: E \rightarrow \{+, -\}$ known as signature or sign function. A vertex signed graph is a triplet $I=(G, \sigma, \gamma)$ where $G=(V,E)$ is the underlying graph and $\sigma: E \rightarrow \{+, -\}$ called sign function and $\gamma: V \rightarrow \{+, -\}$ called vertex sign function. A signed graph $S$ is said to be vertex equitable if it admits a vertex equitable labeling. In this paper, we initiate the vertex equitable labeling of a vertex signed graph and we study the behavior of vertex signed paths through some relevant examples and obtain certain results.

Keywords

Labeling, Vertex Signed Graph, Vertex Equitable Labeling, Vertex Equitable Labeling of Signed Graph, Vertex Equitable Labeling of Vertex Signed Graph.

I. INTRODUCTION

Graph theory said to have its beginning in 1736 when great Swiss mathematician Leonhard Euler considered the Konigsberg bridge problem. In the area of graph theory in mathematics, a signed graph is a graph in which each edge maps to positive or negative sign. Signed graphs are used as effective analytical models for understanding the processes leading to energy crisis, conflict resolution and even in pure mathematical fields such as matrix analysis, root systems, hyper geometry, linear algebra etc. The discovery of signed graph was by the American mathematician Frank Harary as appropriate prototype models to represent structures of cognitive inter relationships in a social group. A signed graph is balanced if it has no half edges and every cycle in it is positive. Balancing concept of signed graphs is widely used in social relations, psychology, international relations and even political science as well.

In literature, many types of graph labeling exist, i.e., graceful labeling, multiplicative labeling, vertex equitable labeling, harmonious labeling, cordial labeling, set labeling and so on. Here we introduce only a vertex equitable labeling in the realm of vertex signed graphs and also we study about the behavior of vertex signed path.
For the terms not mentioned in this paper is due to [1] and whole research work mentioned in this paper is taken from [4], [5], [6].

II. PRELIMINARIES

2.1 Signed Graph

A signed graph \( S = (G, \sigma) \) consists of an underlying graph \( G = (V, E) \) and a function \( \sigma : E \to \{+, -\} \) known as signature or sign function.

2.2 Vertex Signed Graph

A vertex signed graph is a triplet \( \Gamma = (G, \sigma, \gamma) \) in which \( G = (V, E) \) where \( V \) is the set of all vertices, \( E \) is the set of all edges, \( \sigma : E \to \{+, -\} \) is a sign function which assigns either positive or negative sign to every edge in \( E \) and another function \( \gamma : V \to \{+, -\} \) is vertex sign function which assigns either positive or negative sign to every vertex. Positive edges are drawn with solid lines and negative edges are drawn with dotted lines. Half edges or loose edges will not get a sign.

Also positive vertices are marked with shaded circles and negative vertices are marked with open non-shaded circles.

![Fig. 1 A Vertex signed graph \( \Gamma = (G, \sigma, \gamma) \) .](image)

\[
E = \{ e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8 \}
\]

\[
V = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7 \}
\]

The edges \( e_1, e_3, e_5, e_8 \) are positive.
The edges \(e_2, e_4, e_6, e_7\) are negative.

The vertices \(v_1, v_4, v_5\) are positive.

The vertices \(v_2, v_3, v_6, v_7\) are negative.

2.3 Vertex Equitable Labeling

Suppose \(G\) is a \((p, q)\) graph and \(A = \{0, 1, 2 \ldots \lceil \frac{p}{2} \rceil \}\). A vertex labeling \(f: V(G) \rightarrow A\) which is onto is said to be vertex equitable labeling of \(G\) if it induces a bijective edge labeling \(f^*: E(G) \rightarrow \{1, 2 \ldots q\}\) given by \(f^*(uv) = f(u) + f(v)\) such that \(|v_f(a) - v_f(b)| \leq 1\) for all \(a, b \in A\) where \(v_f(a)\) is the number of vertices with \(f(v) = a\). Here \([n]\) denotes the smallest integer greater than or equal to \(n\). Graph \(G\) is vertex equitable if \(G\) admits a vertex equitable labeling.

2.4 Vertex Equitable Signed Graph

Let \(S\) is a \((p, q)\) signed graph with \(q = m + n\) where \(m\) (or \(n\)) is the number of positive (negative) edges in \(S\) and \(A = \{0, 1, 2 \ldots \lceil \frac{p}{2} \rceil \}\). A vertex labeling \(f: V(S) \rightarrow A\) which is onto is said to be vertex equitable labeling of \(S\) if it induces a bijective edge labeling \(f^*: E(S) \rightarrow \{1, 2 \ldots m, -1, -2 \ldots -n\}\) defined by \(f^*(uv) = \sigma(uv)(f(u) + f(v))\) such that \(|v_f(a) - v_f(b)| \leq 1\) for all \(a, b \in A\) where \(v_f(a)\) is the number of vertices with \(f(v) = a\).

Signed graph is said to be vertex equitable if it admits a vertex equitable labeling.

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Fig. 2 A Signed Graph of \(S = (G, \sigma)\).
\( q = \text{Total number of edges} = 6 \)

\( m = 3 = \text{Number of positive edges} \)

\( n = 3 = \text{Number of negative edges} \)

\( A = \{0, 1, 2, \ldots, \left\lfloor \frac{6}{2} \right\rfloor \} = \{0, 1, 2, 3\} \)

\( f : V(S) \rightarrow A \)

\( f : \{v_1, v_2, v_3, v_4, v_5, v_6\} \rightarrow \{0, 1, 2, 3\} \)

\[ v_f(0) = 1 \]
\[ v_f(1) = 2 \]
\[ v_f(2) = 2 \]
\[ v_f(3) = 1 \]

\( \forall a, b \in A \) Provided \( |v_f(a) - v_f(b)| \leq 1 \).

Now, \( f^* : E(S) \rightarrow \{1, 2, 3, -1, -2, -3\} \)

\( f^* : \{e_1, e_2, e_3, e_4, e_5, e_6\} \rightarrow \{1, 2, 3, -1, -2, -3\} \)

\[ f^*(e_1) = \sigma(e_1) (f(v_1) + f(v_4)) = -1 \]

\[ f^*(e_2) = \sigma(e_2) (f(v_2) + f(v_4)) = -2 \]

\[ f^*(e_3) = \sigma(e_3) (f(v_3) + f(v_4)) = -3 \]

\[ f^*(e_4) = \sigma(e_4) (f(v_5) + f(v_4)) = +2 \]
Corresponding vertex equitable signed graph is given below.

\[ f^*(e_5) = \sigma(e_5) (f(v_5) + f(v_6)) = +3 \]
\[ f^*(e_6) = \sigma(e_6) (f(v_6) + f(v_4)) = +1 \]

Fig. 3  A Vertex Equitable Signed Graph of \( S = (G, \sigma) \).

III. MAIN RESULTS

3.1 Vertex Equitable Vertex Signed Graph

Let \( \Gamma \) be a \((p, q)\) vertex signed graph with \( p = r + s \) where \( r, s \) are the number of positive, negative vertices in \( \Gamma \). Also \( q = m + n \) where \( m, n \) are the number of positive, negative edges in \( \Gamma \). The set \( A \) is defined as \( A = \{0, 1, 2, \ldots, \left\lfloor \frac{r}{2} \right\rfloor \} \). A vertex labeling \( f : V(\Gamma) \rightarrow A \) which is onto is said to be vertex equitable labeling of \( \Gamma \) if it induces a bijective edge labeling \( f^* : E(\Gamma) \rightarrow \{1, 2 \ldots m, -1, -2 \ldots -n\} \) defined by,

\[ f^*(uv) = \gamma(u)\gamma(v)\sigma(uv)(f(u) + f(v)) \text{ such that } |v_f(a) - v_f(b)| \leq 1 \text{ } \forall \text{ } a, b \in A \text{ } \text{ where } v_f(a) \text{ is the number of vertices with } f(v) = a. \]
3.2 Example

Fig.4 Vertex Signed Graph $\Gamma = (G,\sigma, \gamma)$

$p = 6$ (Total number of vertices)

$r = 3$ (Number of positive vertices)

$s = 3$ (Number of negative vertices)

$q = 6$ (Total number of edges)

$m = 3$ (Total number of positive edges)

$n = 3$ (Total number of negative edges)

$A = \{0,1,2 \ldots \left\lceil \frac{q}{2} \right\rceil \}$

$= \{0, 1, 2, 3\}$.

$f: V(\Gamma) \rightarrow A$.

$f: \{v_1, v_2 \ldots v_6\} \rightarrow \{0,1,2,3\}$.

$f^*: E(\Gamma) \rightarrow \{-1, -2, -3, 1, 2, 3\}$.
\[ f^*(e_1) = f^*(v_1v_4) \]
\[ = \gamma(v_1) \gamma(v_4) \sigma(e_1) (f(v_1) + f(v_4)) \]
\[ = (+) (-) (+) (1+0) \]
\[ = -1 \]
\[ f^*(e_2) = f^*(v_2v_4) \]
\[ = \gamma(v_2) \gamma(v_4) \sigma(e_2) (f(v_2) + f(v_4)) \]
\[ = (-) (-) (-) (2) \]
\[ = -2 \]
\[ f^*(e_3) = f^*(v_3v_4) \]
\[ = \gamma(v_3) \gamma(v_4) \sigma(e_3) (f(v_3) + f(v_4)) \]
\[ = (+) (-) (+) (3) \]
\[ = -3 \]
\[ f^*(e_4) = f^*(v_4v_5) \]
\[ = \gamma(v_4) \gamma(v_5) \sigma(e_4) (f(v_4) + f(v_5)) \]
\[ = (-) (+) (-) (1) \]
\[ = +1 \]
\[ f^*(e_5) = f^*(v_5v_6) \]
\[ = \gamma(v_5) \gamma(v_6) \sigma(e_5) (f(v_5) + f(v_6)) \]
\[ = (+) (-) (-) (3) \]
\[ = +3 \]
\[ f^*(e_6) = f^*(v_6v_4) \]
\[ = \gamma(v_6) \gamma(v_4) \sigma(e_6) (f(v_6) + f(v_4)) \]
\[ = (-) (-) (+) (2) \]
\[ = +2. \]

The mapping \( f^*: \{e_1, e_2, e_6\} \rightarrow \{-3, -2, -1, 1, 2, 3\} \) is one to one and onto. Hence this graph is vertex equitable, since it has a vertex equitable labeling. Corresponding vertex equitable vertex signed graph is given below.
Fig. 5 Vertex Equitable Vertex Signed Graph of $\Gamma = (G, \sigma, \gamma)$

### 3.3 Homogeneous Vertex Signed Graphs

A vertex signed graph with all positive vertices and edges or all negative vertices and edges is known as homogeneous vertex signed graph.

### 3.4 Example

Fig. 6 Homogeneous Vertex Signed Graph (All negative vertices and edges).
3.5 Heterogeneous Vertex Signed Graph

If a vertex signed graph is not homogeneous we call it as heterogeneous.

3.6 Vertex Signed Path

Vertex signed path $P_n$ is a path with $n$ vertices such that every edge and every vertex will get a sign (Positive or Negative).

3.7 Example

![Vertex Signed Path $P_5$](image)

3.8 Negation of a Vertex Signed Graph

The negation of a vertex signed graph $\Gamma$ is denoted by $\rho(\Gamma)$, negating the sign of every edge and vertex of $\Gamma$. That is changing the sign of every edge and vertex to its opposite.
3.9 Homogeneous Vertex Signed Path

A vertex signed path with either all positive vertices or edges or all negative vertices and edges is homogeneous vertex signed path.

3.10 Theorem

Homogeneous vertex signed path $P_n$ is vertex equitable.

Proof

Let $P_n$ be a homogeneous vertex signed path with $n$ vertices and $n-1$ edges. And $A = \{0, 1, 2 \ldots \lceil \frac{n}{2} \rceil \}$. There are two cases need to be considered.

Case 1: $P_n$ has all vertices and all edges with a positive sign. In this case, the mapping $f^*$ will be a mapping from the edge set $E(P_n)$ to $\{1, 2 \ldots n-1\}$. Now we define,

$$f : V(P_n) \rightarrow \{0, 1, 2 \ldots \lceil \frac{n}{2} \rceil \} \text{ as } f(v_1) = 1, f(v_2) = 0, f(v_3) = 2, f(v_i) = f(v_{i-2}) + 1 \text{ where } 4 \leq i \leq n.$$ 

Now it is very clear that $f$ is a vertex equitable labeling.

Case 2: $P_n$ has all edges and vertices as negative. In this case $f^*$ will be a mapping from $E(P_n) \rightarrow \{-1, -2 \ldots -n\}$ and then we can use the same definition of function as above. Since the multiplication of two negative vertices and one negative edge is being negative we obtain the required vertex equitable labeling. Hence the proof.

3.11 Example

Consider $P_6$, then the number of vertices=6, number of edges=5.

And $A = \{0, 1, 2 \ldots \lceil \frac{6}{2} \rceil \} = \{0, 1, 2, 3\}$. 

![Diagram of vertex signed path](attachment://image.png)
\[ f^*: E(P_n) \rightarrow \{1,2,3,4,5\} \]
\[ f^*(e_1)= f^*(v_1v_2)=1,\text{similarly } f^*(e_2) = 2, f^*(e_3) = 3, f^*(e_4) = 4, f^*(e_5) = 5. \]

Hence it is vertex equitable.

**3.12 Corollary**

A vertex signed path \( P_n \) having a negative pendant edge and all other edges and vertices being positive is vertex equitable.

**3.13 Corollary**

Every vertex signed path with all negative vertices and all positive edges is vertex equitable.

**3.14 Corollary**

Every vertex signed path with all positive vertices and all negative edges is vertex equitable.

**IV. CONCLUSIONS**

In this paper, we introduced the concept of vertex equitable labeling of vertex signed graphs through some relevant example. Also, we have defined and studied about homogeneous vertex signed path as well as heterogeneous vertex signed path.

**REFERENCES**


[6] Thomas Zaslavsky, Signed graphs and geometry, Department of Mathematical Sciences, Binghamton University, Binghamton, NY13902-6000, U.S.A.
